Lecture 4:

Recap:

Image decomposition based on Singular Value Decomposition (SVD)

Definition: (SVD) For any $g \in \mathbb{R}^{m \times n}$, the singular value decomposition (SVD) of g is a matrix factorization: $g = U \Sigma V^T$, where $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ are unitary, Σ is a diagonal matrix ($\Sigma_{ij} = 0$ if $i \neq j$) with diagonal entries given by: $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ with $r \leq \min(m, n)$.

Remark: r = rank of g !!

How to compute SVD

Let $A \in M_{m\times n}$ $(m \geq n)$ $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$

Step 1: Find eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ $\rightarrow V$

of symmetric $A^TA \in M_{n\times n}$ (with $\|\vec{v}_j\| = 1$, $j = 1, \dots, n$)

[Recall: $(A^TA)\vec{v}_j = \lambda_j \vec{v}_j$]

Step 2: Define:

$$\Sigma = \left(\begin{array}{cccc} \overset{=\sigma_1}{\sqrt{\lambda_1}} & & & \\ & \overset{=\sigma_2}{\sqrt{\lambda_2}} & & \\ & & \ddots & \\ & & & \overset{=\sigma_n}{\sqrt{\lambda_n}} \\ \hdashline & & 0 & \end{array}\right) \in M_{m\times n}$$

Add zero rows if $m > n$

Step 3: For non-zero $\sigma_1, \sigma_2, \dots, \sigma_r$,

let $\vec{u}_1 = \dfrac{A\vec{v}_1}{\sigma_1}$, $\vec{u}_2 = \dfrac{A\vec{v}_2}{\sigma_2}$, $\dots$, $\vec{u}_r = \dfrac{A\vec{v}_r}{\sigma_r}$

Step 4: Extend $\{\vec{u}_1, \dots, \vec{u}_r\}$ to the basis $\{\vec{u}_1, \dots, \vec{u}_r, \dots, \vec{u}_m\}$ of $\mathbb{R}^m$. $\rightarrow U$

Step 5: Let:

$$U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{pmatrix} \in M_{m\times m}$$

$$V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}$$

Then: $A = U\Sigma V^T$

Example 2.1: Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

We have

$$A^T A = \begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix}.$$

(Step 1)

$$\frac{A\vec{v}_2}{\sigma_2}$$

Now, eig(A^TA) are 17 and 1, and so $\sigma_1 = \sqrt{17}$, $\sigma_2 = 1$ and

$$\Sigma = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover,

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$u_i = \frac{A\vec{v}_i}{\sigma_i}$$

Since

$$\sigma_1 \vec{u}_1 = A\vec{v}_1,$$

we have

$$\vec{u}_1 = \underset{\sigma_1}{\frac{1}{\sqrt{17}}} \frac{1}{\sqrt{2}} \overbrace{\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}}^{A} \underset{\vec{v}_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \frac{1}{\sqrt{34}}\begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}.$$

Similarly, we have

$$\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The matrix U is, therefore, given by

$$U = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \\ \frac{4}{\sqrt{34}} & 0 & \mathbf{u}_3 \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \end{pmatrix}$$

for some vector $\mathbf{u}_3$ orthonormal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. One possibility is

$$\vec{u}_3 = \frac{1}{\sqrt{17}}\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}.$$

Finally, the SVD of A is given by

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix}\begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

Remark:

1. Note that $gg^T = U\Lambda^{1/2}\underbrace{V^TV}_{I}\Lambda^{1/2}U^T = U\Lambda U^T$

$\therefore U$ consists of eigenvectors of gg^T.

Note that $g^Tg = V\Lambda^{1/2}\underbrace{U^TU}_{I}\Lambda^{1/2}V^T = V\Lambda V^T$

$\therefore V$ consists of eigenvectors of g^Tg.

2. Note that $g = U\underline{\underline{\Lambda^{1/2}}}V^T = \displaystyle\sum_{i=1}^{r}\sigma_i\, U \begin{array}{cl} \begin{array}{c}\text{ith}\\ \downarrow\end{array} & \\ \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} & \leftarrow \text{ith} \end{array} V^T = \displaystyle\sum_{i=1}^{r}\sigma_i\,\vec{u}_i\,\vec{v}_i^{\,T}$

$\vec{u}_i\,\vec{v}_i^{\,T}$ is called the eigen-image of g under SVD.

3. For $N\times N$ image, the required storage is:

$$(\underbrace{N}_{\vec{u}_i} + \underbrace{N}_{\vec{v}_i} + \underbrace{1}_{\sigma_i}) \times \underbrace{r}_{r\text{ terms}} = (2N+1)r$$

Definition: For any k $(0 \le k \le r)$, we define

$$g_k = \sum_{j=1}^{k}\sigma_j\,\vec{u}_j\,\vec{v}_j^{\,T} \quad (\text{rank-}k \text{ approximation of } g)$$

Error of the approximation by SVD

Theorem: Let $f = \sum_{j=1}^{r} \sigma_j \vec{u}_j \vec{v}_j^T$ be the SVD of a $M \times N$ image f. For any $k < r$,

and $f_k = \sum_{j=1}^{k} \sigma_j \vec{u}_j \vec{v}_j^T$, we have: $\| f - f_k \|_F^2 = \sum_{i=k+1}^{r} \underbrace{\sigma_i^2}_{\lambda_i}$

Proof: Let $f = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$.

Let $D \equiv f - f_k = \sum_{i=k+1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T \in M_{M \times N}$.

Then, the m-th row, n-th col entry of D is given by:

$$D_{mn} = \sum_{i=k+1}^{r} \sigma_i u_{im} v_{in} \in \mathbb{R} \quad \text{where} \quad \vec{u}_i = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iM} \end{pmatrix} ; \quad \vec{v}_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{iN} \end{pmatrix}$$

$$\therefore D_{mn}^2 = \left(\sum_{i=k+1}^{r} \sigma_i u_{im} v_{in} \right)^2 = \sum_{i=k+1}^{r} \sigma_i^2 u_{im}^2 v_{in}^2 + 2 \sum_{i=k+1}^{r} \sum_{\substack{j=k+1 \\ j \neq i}}^{r} \sigma_i \sigma_j u_{im} v_{in} u_{jm} v_{jn}.$$

Thus, $\|D\|_F^2 = \sum_m \sum_n D_{mn}^2$

$$= \sum_m \sum_n \left(\sum_{i=k+1}^{r} \sigma_i^2 u_{im}^2 v_{in}^2 + 2 \sum_{i=k+1}^{r} \sum_{\substack{j=k+1 \\ j \neq i}}^{r} \sigma_i \sigma_j u_{im} v_{in} u_{jm} v_{jn} \right)$$

$$= \sum_{i=k+1}^{r} \sigma_i^2 \overbrace{\sum_m u_{im}^2}^{1} \overbrace{\sum_n v_{in}^2}^{1} + 2 \sum_{i=k+1}^{r} \sum_{\substack{j=k+1 \\ j \neq i}}^{r} \sigma_i \sigma_j \overbrace{\sum_m u_{im} u_{jm}}^{0} \overbrace{\sum_n v_{in} v_{jn}}^{0}$$

$$= \sum_{i=k+1}^{r} \sigma_i^2 \qquad \sigma_i^2 = \lambda_i$$

Remark:

- To approximate an image using SVD, arrange the eigenvalues λ_i in decreasing order and remove the last few terms in $\sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$
- rank - k approximation is the optimal approximation using k-terms (in term of F-norm) (or with rank-k image)

Theorem: (Existence of SVD) Every $m \times n$ image has a SVD.

Proof: Consider the case when $m \leq n$.

We need the following theorem.

Theorem: Let $B \in M_{n\times n}$ be a real symmetric matrix. Then, $\exists$ orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ with corresponding eigenvalues such that

$$B = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{pmatrix}$$

Note that $gg^T \in M_{m\times m}$ and $g^Tg \in M_{n\times n}$ are symmetric.

$\therefore \exists$ n pairwise orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ of g^Tg.

Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ are associated with non-zero eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$.

Note that $gg^T(g\vec{v}_i) = g(\lambda_i \vec{v}_i) = \lambda_i (g\vec{v}_i)$.

$\therefore$ $g\vec{v}_i$ is an eigenvector of gg^T with eigenvalue λ_i.

Let $\sigma_i = \sqrt{\lambda_i}$. Then: $\|g\vec{v}_i\|^2 = (g\vec{v}_i)^T(g\vec{v}_i) = \vec{v}_i^T g^T g \vec{v}_i = \vec{v}_i^T(\lambda_i \vec{v}_i) = \lambda_i$.

$\therefore$ $\|g\vec{v}_i\| = \sigma_i$

Define $\vec{u}_i = \dfrac{g\vec{v}_i}{\sigma_i}$. Then: $\|\vec{u}_i\| = 1$.

$$= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Also, $$\vec{u}_i \cdot \vec{u}_j = \frac{(g\vec{v}_i)^T(g\vec{v}_j)}{\sigma_i \sigma_j} = \frac{\vec{v}_i^T g^T g \vec{v}_j}{\sigma_i \sigma_j} = \frac{\lambda_j \vec{v}_i^T \vec{v}_j}{\sigma_i \sigma_j} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\vec{u}_i \cdot g\vec{v}_j = \sigma_j \vec{u}_i^T \vec{u}_j = \begin{cases} \sigma_i & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

In matrix form,
$$\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \end{pmatrix}}_{r \times m} \underbrace{g}_{m \times n} \underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ | & | & & | \end{pmatrix}}_{n \times r} = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{pmatrix}$$

Extend $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$ to an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_r, \vec{u}_{r+1}, \dots, \vec{u}_m\}$ of $\mathbb{R}^m$.

Extend $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ to an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$ of $\mathbb{R}^n$.

Then:

$$\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \\ & \vdots & \\ - & \vec{u}_m^T & - \end{pmatrix}}_{m \times m} \underbrace{g}_{m \times n} \underbrace{\begin{pmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_r & \cdots & \vec{v}_n \\ | & & | & & | \end{pmatrix}}_{n \times n} = \underbrace{\begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}}_{m \times n} \neq \Lambda^{1/2}$$

(Here, we need to use the fact that $g\vec{v}_j = 0$ for $j > r$, since $\|g\vec{v}_j\| = \sigma_j = 0$ for $j > r$)

Note that $U^T U = U U^T = I$; $V^T V = V V^T = I$. $\therefore$ $g = U \Lambda^{1/2} V^T$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

(The case for $m > n$ can be shown similarly)

Recap on the proof of existence:

Theorem: (Existence of SVD) Every $m \times n$ image has a SVD.

Proof: Consider the case when $m \leq n$.

We need the following theorem.

Theorem: Let $B \in M_{n \times n}$ be a real symmetric matrix. Then, $\exists$ orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with corresponding eigenvalues such that

$$B = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{pmatrix}$$

Note that $gg^T \in M_{m \times m}$ and $g^Tg \in M_{n \times n}$ are symmetric.

$\therefore \exists$ n pairwise orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ of g^Tg.

Find orthonormal basis of g^Tg

Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are associated with non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$.

Note that $gg^T(g\vec{v}_i) = g(\lambda_i \vec{v}_i) = \lambda_i (g\vec{v}_i)$.

$\therefore$ $g\vec{v}_i$ is an eigenvector of gg^T with eigenvalue λ_i.

Note that g^Tg is positive-definite and hence all of its eigenvalues must be +ve.

$\therefore$ $\lambda_i > 0$ for $i = 1, 2, \dots, r$.

Let $\sigma_i = \sqrt{\lambda_i}$. Then: $\|g\vec{v}_i\|^2 = (g\vec{v}_i)^T(g\vec{v}_i) = \vec{v}_i^T g^T g \vec{v}_i = \vec{v}_i^T(\lambda_i \vec{v}_i) = \lambda_i$.

Define $\vec{u}_i$

$\therefore$ $\|g\vec{v}_i\| = \sigma_i$

$$\boxed{\text{Define } \vec{u}_i = \frac{g\vec{v}_i}{\sigma_i}}$$

Then: $\|\vec{u}_i\| = 1$.

$$\begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Also, $$\vec{u}_i \cdot \vec{u}_j = \frac{(g\vec{v}_i)^T(g\vec{v}_j)}{\sigma_i \sigma_j} = \frac{\vec{v}_i^T g^T g \vec{v}_j}{\sigma_i \sigma_j} = \frac{\lambda_j \vec{v}_i^T \vec{v}_j}{\sigma_i \sigma_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\vec{u}_i \cdot g\vec{v}_j = \sigma_j \vec{u}_i^T \vec{u}_j = \begin{cases} \sigma_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

In matrix form,

$$\boxed{\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \end{pmatrix}}_{r \times m} \underbrace{g}_{m \times n} \underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ | & | & & | \end{pmatrix}}_{n \times r} = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{pmatrix}}$$

Form preliminary matrix decomposition

Extend $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ to an orthonormal basis $\{\vec{u}_1, \ldots, \vec{u}_r, \vec{u}_{r+1}, \ldots, \vec{u}_m\}$ of $\mathbb{R}^m$.

Extend $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ to an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_r, \vec{v}_{r+1}, \ldots, \vec{v}_n\}$ of $\mathbb{R}^n$.

Extend basis

Then:

$$\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \\ & \vdots & \\ - & \vec{u}_m^T & - \end{pmatrix}}_{m\times m} \underbrace{g}_{m\times n} \underbrace{\begin{pmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_r & \cdots & \vec{v}_n \\ | & & | & & | \end{pmatrix}}_{n\times n} = \underbrace{\begin{pmatrix} \sigma_1 & & & & & & \\ & \sigma_2 & & & & & \\ & & \ddots & & & & \\ & & & \sigma_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}}_{m\times n} = \Lambda^{1/2}$$

(Here, we need to use the fact that $g\vec{v}_j = 0$ for $j > r$, since $\|g\vec{v}_j\| = \sigma_j = 0$ for $j > r$)

Note that $U^TU = UU^T = I$; $V^TV = VV^T = I$. $\therefore$ $g = U\Lambda^{1/2}V^T$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

(The case for $m > n$ can be shown similarly)

Haar transformation

Definition: (Haar functions) The Haar functions are defined recursively as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

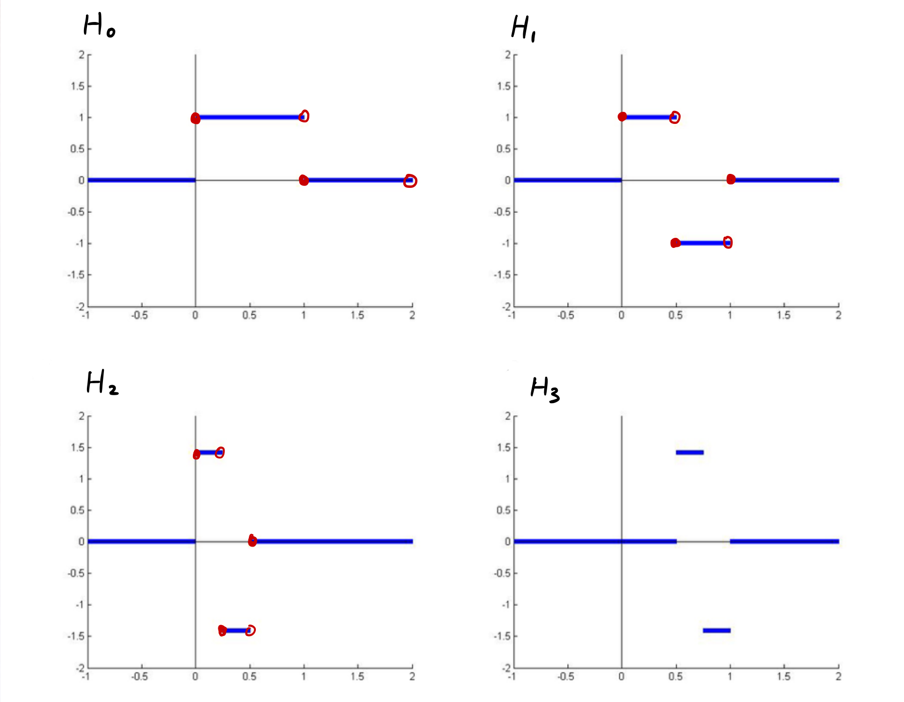
$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^{\,p} & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^{\,p} & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \dots;\ n = 0, 1, 2, \dots, 2^p - 1$

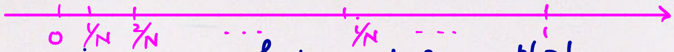
Remark: If p is larger, H_{2^p+n} is compactly supported in a smaller region.

Examples of Haar functions:



Definition (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is done by dividing $[0,1]$ into partitions.



Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$.

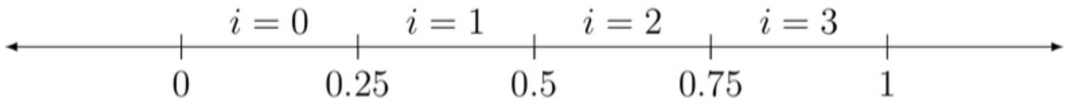
We obtain the Haar Transform matrix: $\tilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv \left(H(k,i)\right)_{0 \leq k, i \leq N-1}$

The Haar Transform of $f \in M_{n \times n}$ is defined as:

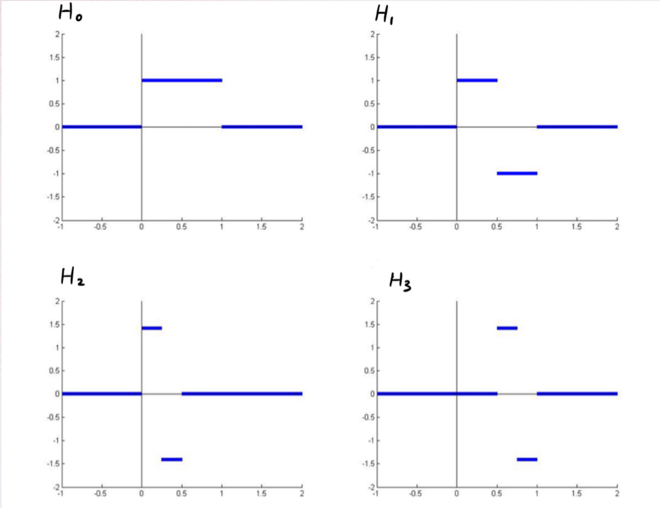
$$g = \tilde{H} f \tilde{H}^T.$$

Example Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0,1]$ into 4 portions:



Need to check:



We get that:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{4}} H = \frac{1}{2} H$$

Easy to check that $\tilde{H}^T \tilde{H} = I$.

Example 2 Compute the Haar Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{H} f \tilde{H}^T = \left.\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\right\} \text{More zeros}$$

Example 3 Suppose g in Example 2 is changed to:

$$g = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Reconstruct the original image.

Solution:

$$f = \tilde{H}^T f \tilde{H} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Localized error

Remark:

1. Haar Transform usually produces coefficient matrix with more zeros!

2. Localized error in coefficient matrix causes localized error in the reconstructed image

Elementary images under Haar transform:

Using Haar transform, f can be written as:

$$f = \tilde{H}^T \underset{\uparrow}{g} \tilde{H}$$

transformed image

Let $\tilde{H} = \begin{pmatrix} - \vec{h}_1^T - \\ - \vec{h}_2^T - \\ \vdots \\ - \vec{h}_N^T - \end{pmatrix}$. Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{h}_i \vec{h}_j^T}_{= I_{ij}^H}$

I_{ij}^T = elementary images under Haar Transform.

Definition: (Walsh function) The Walsh functions are defined recursively by:

$$W_{2j+q}(t) \equiv (-1)^{\lfloor \frac{j}{2} \rfloor + q} \left\{ W_j(2t) + (-1)^{j+q} W_j(2t-1) \right\}$$

where $\lfloor \frac{j}{2} \rfloor$ = biggest integer smaller than or equal to $\frac{j}{2}$.

$q = 0$ or 1, $j = 0, 1, 2, \dots$ and

$$W_0(t) \equiv \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

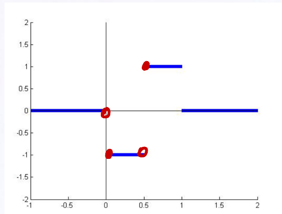
Example: Compute $W_1(t)$.

Put $j = 0$, $q = 1$. Then:

$$W_1(t) = (-1)^{\lfloor 0 \rfloor + 1} \left\{ W_0(2t) + (-1)^1 W_0(2t-1) \right\} = (-1)\left\{ W_0(2t) + (-1)^1 W_0(2t-1) \right\}$$

For $0 \le t < \frac{1}{2}$, $W_0(2t) = 1$, $W_0(2t-1) = 0$ $\Rightarrow W_1(t) = -1$.

For $\frac{1}{2} \le t < 1$, $W_0(2t) = 0$, $W_0(2t-1) = 1$ $\Rightarrow W_1(t) = 1$.



Definition: (Discrete Walsh transform)

The Walsh Transform of a $N \times N$ image is defined as follows.

Define $W(k,i) \equiv W_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$.

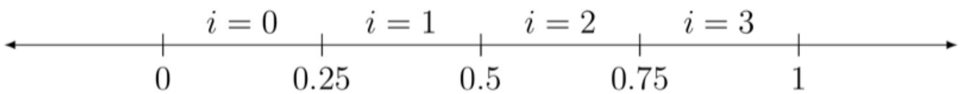
The Walsh transform matrix is: $\tilde{W} \equiv \frac{1}{\sqrt{N}} W$ where $W \equiv \left(W(k,i)\right)_{0 \le k, i \le N-1}$

The Walsh transform of $f \in M_{n \times n}$ is defined as:

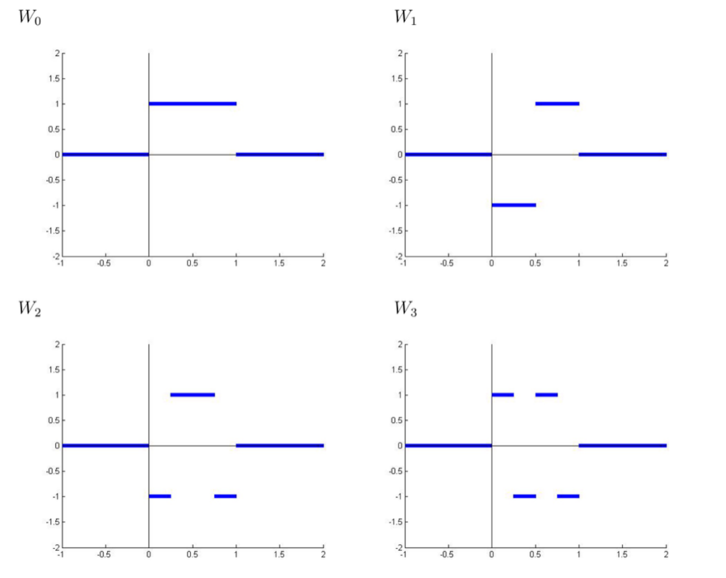
$$g = \tilde{W} f \tilde{W}^T$$

Example Compute the Walsh Transform matrix for a 4×4 image.

Solution: Again, divide $[0, 1]$ into 4 portions:



We can check that:



So,

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \tilde{W} = \frac{1}{\sqrt{4}} W = \frac{1}{2} W$$

$(\tilde{W}^T \tilde{W} = I)$

Example 2.7: Compute the Walsh Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{W} f \tilde{W}^T = \left.\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\} \text{More zeros in the coefficient matrix!}$$

Remark: 1. Walsh transform is to transform an image to a "transformed image" with much more zeros.

Elementary images under Walsh transform:

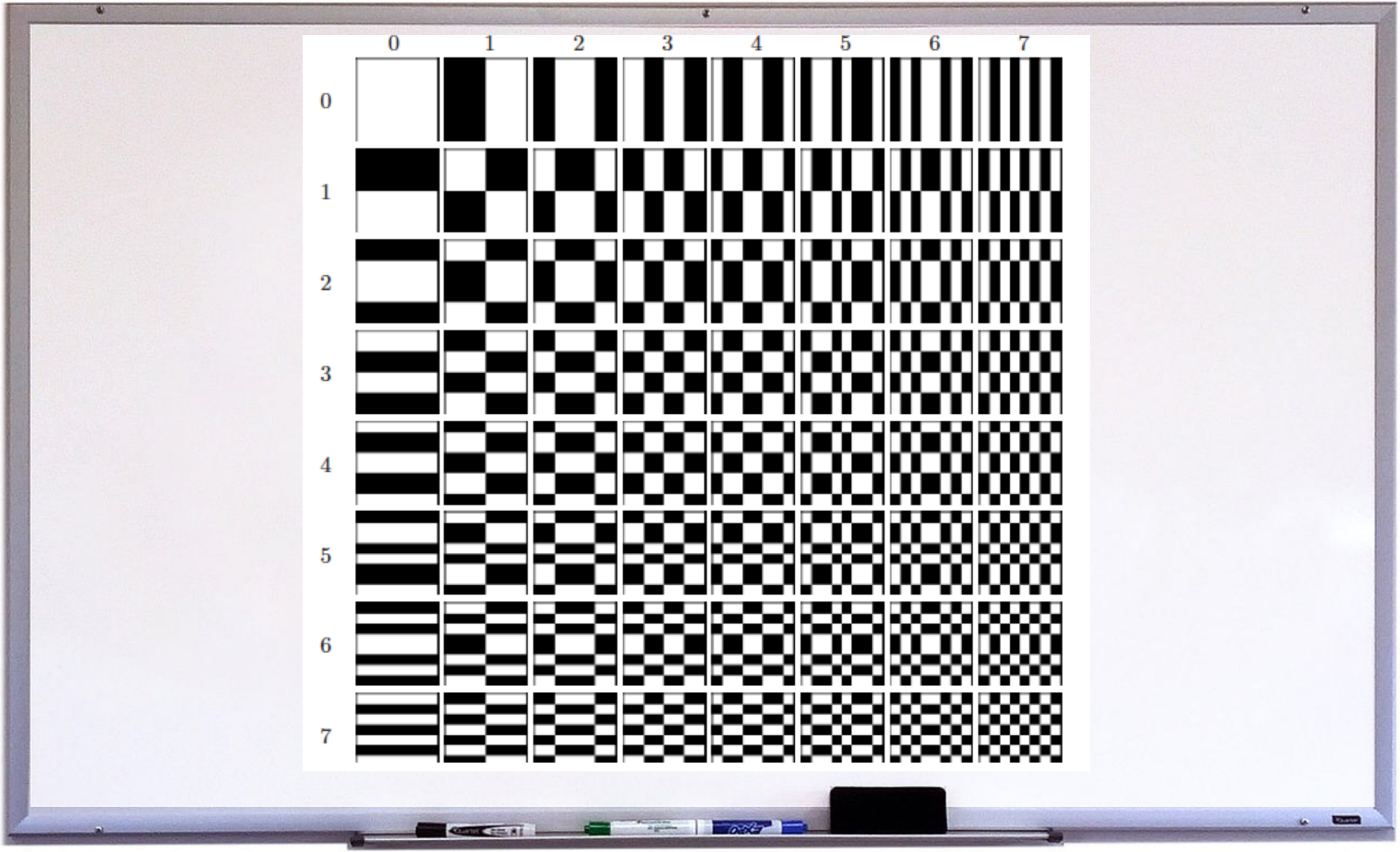
Under Walsh Transform, $f = \tilde{W}^T g \tilde{W}$.

transformed image

Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \vec{w}_i \vec{w}_j^{\,T}$ where $\tilde{W} = \begin{pmatrix} - & \vec{w}_1^{\,T} & - \\ - & \vec{w}_2^{\,T} & - \\ & \vdots & \\ - & \vec{w}_N^{\,T} & - \end{pmatrix}$

" I_{ij}^{W}

I_{ij}^{W} = elementary images under Walsh transform.


0 1 2 3 4 5 6 7
0
1
2
3
4
5
6
7